1.

i. Find the solution of the differential equation

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+13y=0$$

for which $y=0$ and $\frac{dy}{dx}=1$ when $x=0$.

ii. Given that

$$\cos x\frac{dy}{dx}+2y\sin x=\cos^3 x+\sin x\,,0<x<\frac{\pi}{2}$$

and $y=1$ when $x=\frac{1}{3}\pi$, find the value of y when $x=\frac{1}{4}\pi$.

2. Given the function f, where

$$f(x)=\frac{x^2+x+2}{x-1}$$

.

a. Show that for real x, $f(x)$ cannot take values between -1 and 7.

b. Find the equations of the two asymptotes to the curve $y=f(x)$.

c. Sketch the curve $y=f(x)$, showing clearly the turning points, intercepts, coordinate axes and how the curve approaches the asymptotes.

3. A transformation T of three-dimensional space is defined by $\boldsymbol{Mr}=\boldsymbol{r'}$, where

$$\boldsymbol{M}=\begin{pmatrix}7&5&6\\4&3&3\\10&7&k\end{pmatrix},\qquad \boldsymbol{r}=\begin{pmatrix}x\\y\\z\end{pmatrix},\qquad \boldsymbol{r'}=\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$$

and k a real constant.

a. Find the value of k for which there is no inverse transformation.

b. If $k=9$, show that (x,y,z) are transformed into the plane $2x'-y'-z'=0$.

c. If $k=8$, find $\boldsymbol{M}^{-1}$, and hence, or otherwise, find the point which is mapped onto $(6,2,9)$.

4. A curve is parametrically defined by $x=a\left(\frac{1}{3}t^3-t\right);y=at^2$. The points A and B on the curve are given by $t=0$ and $t=3$ respectively.

a. Show that the length of the arc AB is $12a$.

b. Show further that the surface area generated by one complete revolution of the art AB about the x −axis is given by $\frac{576}{5}\pi a^2$.

c. Use the theorem of Pappus to find the y −coordinate of the centroid of the arc AB.

5.

i. By using the substitution $u=1+\cosh x$, evaluate

$$\int_0^{\cosh^{-1}2}\frac{\tanh x}{1+\cosh x}dx$$

leaving your answer in terms of natural logarithms.

ii. Show that

$$\frac{d}{dx}(\tanh^{-1}x)=\frac{1}{1-x^2}$$

. Hence, or otherwise, show that

$$\int_0^{\frac{1}{2}}\tanh^{-1}x\,dx=\frac{1}{4}\ln\left(\frac{27}{16}\right)$$

.

6.

i.

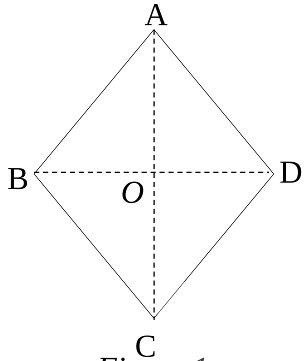

Figure 1

Figure 1 shows a rhombus ABCD. In the set of symmetry transformations of the rhombus, let I denote the identity transformation, H denote reflection in the diagonal BD, V denote reflections in the diagonal AC, R denote the rotation of every point of every point through 180^0 about the point O form the composition elements of the set $G = \{I, H, V, R\}$. Deduce that these elements form a group.

ii. Given the set of matrices $S = \{A, B, C, D\}$ where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

show that S forms a group under matrix multiplication.

State whether or not the groups in (i) and (ii) above are isomorphic and justify your answer. (Assume associativity in each case).

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7. Prove that the tangent at the point $(at^2, 2at)$ on the parabola $y^2 = 4ax$ is $x - ty + at^2 = 0$.
The points A and B on this parabola are given by $A(at_1^2, 2at_1), B(at_2^2, 2at_2)$. Show that the equation of the chord AB is given by $2x - (t_1 + t_2)y + 2at_1t_2 = 0$.
The tangents at A and B meet at the point P. Find the coordinates of P. The line through P, parallel to the axis of the parabola meets the chord AB at M. Find the coordinate of M. Prove that M is the midpoint of AB.

8.

i. In an Argand diagram, shade the region in which the points representing the complex number z can lie, if $|z - 1 + 2i| < |z - 1|$.

ii. Use De Moivre"s theorem to show that $\cos 5\theta = \cos\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$ and that $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$. Hence, prove that

$$\tan 5\theta = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4},$$

where $t = \tan\theta$.

iii. A transformation T in the complex plane is given by the equation $\omega = \frac{iz+1}{z+1}$. Show that the circle $|z| = 2$ is mapped onto the circle $|\omega + 1| = 2|\omega - i|$.

9.

i. The curves C_1 and C_2 are defined by the polar equations:

$$C_1: r = 2\sec\theta, -\frac{\pi}{2} < 0 < \frac{\pi}{2}; C_2: r = \frac{6}{1 + \cos\theta}, -\pi < 0 < \pi$$

Sketch the curves C_1 and C_2 on the same diagram.

a. State the polar coordinates of the point of intersection of C_1 and C_2 on the same diagram.

b. State the polar coordinates of the point of intersection of C_1 and C_2 with the initial lines and the half lines $\theta = \frac{\pi}{2}$ and $\theta = -\frac{\pi}{2}$.

c. Find the polar coordinates of the point of intersection of C_1 and C_2.

ii. Given that $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\, dx$, show that for $n \geq 2, I_n + I_{n-2} = \frac{1}{n-1}$.

END

1.

i. Find the general solution to the differential equation

$$x\frac{dy}{dx} - y = x^2$$

.

ii. Find the value of the constant a such that $y = axe^{-x}$ is a solution of the differential equation

$$2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 2e^{-x}$$

.

Find the solution of this differential equation for which $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$.

2.

a. Express into partial fractions

$$f(x) = \frac{1}{(x^2+1)(x-1)^2}$$

.

b. Hence, evaluate $\int_2^3 f(x)\,dx$.

c. Given that

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n\theta\, d\theta,$$

show that for $n \geq 2$, $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$. Hence, evaluate I_5.

3.

a. Using the definition of $\sinh x$ in terms e^x, show that $\sin^{-1} x = \ln(x + \sqrt{1+x^2})$.
Hence or otherwise, show that

$$\int_0^4 \frac{1}{\sqrt{9x^2+4}}\,dx = \frac{1}{3}\ln(6+\sqrt{37})$$

b. Show that

$$\int_0^{\frac{1}{4}} \tanh^{-1} 2x\,dx = \frac{1}{8}\ln\left(\frac{27}{16}\right).$$

4.

i. Determine whether or not the following series converge:

a. $\sum_{r=0}^{\infty} \frac{r^r}{r!}$; $\left(\text{Hint: } \lim_{r\to\infty}\left(1+\frac{1}{r}\right)^r = e\right)$

b. $\sum_{r=0}^{\infty} \left(\frac{3}{2}\right)^{r^2}$

c. $\sum_{r=1}^{\infty} \left(\frac{r}{2^r}\right)$

ii. Given that the terms in x^5 and higher powers of x may be neglected, show that

$$e^{\cos^2 x} \approx e\left(1 - x^2 + \frac{5}{6}x^4\right).$$

5.

i. Find graphically or otherwise, the set of values of x foe which $|2x - 5| + |x + 2| > 7$.

ii. Given that

$$f(x) = \frac{x^2 - 5x + 6}{x - 1},$$

a. Find the equation of the two asymptotes to the curve $y = f(x)$.

b. Sketch the curve $y = f(x)$, showing clearly the intercepts, asymptotes and turning points.

6.

i. Show that $1 + i$ is a root of the equation $z^4 + 2z^3 - z^2 - 2z + 10 = 0$.
Hence or otherwise, find the remaining roots of the equation.

ii. If $z = \cos\theta + i\sin\theta$, show that:

a. $z + \frac{1}{z} = 2\cos\theta$.

b. $z^n + \frac{1}{z^n} = 2\cos n\theta$.

Hence, or otherwise, show that $32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$. Evaluate

$$\int_0^{\frac{\pi}{12}} (32\cos^6\theta - 15\cos 2\theta)\, d\theta.$$

7.

i. Use the theorems of Pappus to calculate:

a. The volume and,

b. The surface area, of the solid generated when the region for which $x^2 + (y - 9)^2 \leq 9$ is rotated through 2π radians about the x −axis.

ii. A curve is given parametrically $x = \theta - \sin\theta$, $y = 1 - \cos\theta$.

a. Show that the length of the curve, for $0 \leq \theta \leq 2\pi$, is 8.

b. If the arc in (a) is rotated through one complete revolution about the $x -$ axis, show that the surface generated is $\frac{64}{3}\pi$.

8.

a. Find the equation of the normal at the point $P(4, 1)$ to the rectangular hyperbola $xy = 4$.
This normal meets the hyperbola at the point Q. Find the length of PQ.

b. Write down the equations of the two asymptotes to the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$.
The tangent to this hyperbola at the point $P(3\sec\theta, 4\tan\theta)$ meets the asymptotes at S and T. Show that P is the midpoint of ST.

9.

i. A binary operation $*$ is defined on $\mathbb{R}$, the set of real numbers, by $a * b = a + b + ab$. Determine whether or not $(\mathbb{R}, *)$ forms a group.

ii. Define a mapping from $(\mathbb{R}, +)$ to $(\mathbb{R}, +)$ by $f(x) = 3x$, where $\mathbb{R}$ is the set of real numbers.

a. Show that f is a homomorphism.

b. Show, also, that f is an isomorphism.

CGCEB – Further Mathematics Paper 2, JUNE 2010

1.
i. Solve the differential equation

$$(x+2)\frac{dy}{dx} - y = (x+2)^2\,,$$

given that when $x = 0$, $y = -4$.

ii. Given that $y = Axe^x + Bxe^{2x}$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x + 2e^{2x},$$

find the value of the constants A and B.

Hence solve completely the differential equation given that when $y = 0, \frac{dy}{dx} = 1$ when $x = 0$.

2. The position vectors of the points $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$ with respect to the origin $\boldsymbol{O}$ are $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ respectively, where

$$\boldsymbol{a} = 7\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}, \qquad \boldsymbol{b} = \boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k}, \qquad \boldsymbol{c} = \boldsymbol{i} + \boldsymbol{j} - 4\boldsymbol{k}, \qquad \boldsymbol{d} = 2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}.$$

Find:

a. The Cartesian equation of the plane $\boldsymbol{ABC}$.
b. The Cartesian equation of the plane $\boldsymbol{BCD}$.
c. The cosine of the acute angle between the planes $\boldsymbol{ABC}$ and $\boldsymbol{BCD}$.
d. The area of the triangle $\boldsymbol{BCD}$.
e. The volume of the tetrahedron $\boldsymbol{ABCD}$.

3. Prove that the equation of the normal to the rectangular hyperbola $xy = c^2$ at the point $P\left(ct, \frac{c}{t}\right)$ is $t^3x - ty = c(t^4 - 1)$.
The normal at P on the hyperbola meets the $x-$axis at Q and the tangent T meets the $y-$ axis R. Show that the locus of the midpoint of QR, as P varies is $2c^2xy + y^4 = c^4$.

4.
a. Find the root mean square value of $\tanh x$, for $0 \le x \le 2$.
b. A curve is given parametrically by $x = \cosh^2 t\,, y = \sinh^2 t\,, 0 \le t \le 2$.
   i. Find the length of the curve, leaving your answer in terms of e.
   ii. Prove that the area of surface generated by rotating the curve through 2π radians about the $x-$axis is given by $\frac{\pi}{3e^6}\left((e^4+1)^3 - 8e^6\right)$.

5.
a. Prove that the set of numbers $\{1, 2, 4, 5, 7, 8\}$ forms an Abelian group under multiplication modulo 9.
b. Prove also that the set of numbers $\{1, 2, 4, 5\}$ forms an Abelian group under addition modulo 6.
   Are the two groups isomorphic? Give a reason to justify your answer.

6.
a. Test each of the following series for convergence.

$i.\ \displaystyle\sum_{n=0}^{\infty}\left(\frac{2^n+5}{3^n}\right)$

$ii.\ \displaystyle\sum_{n=1}^{\infty}\frac{n-1}{2n^2(n+1)}$

$iii.\ \displaystyle\sum_{n=4}^{\infty}\frac{\sqrt{n}}{n-3}$

b. Find the first three terms of the Taylor series expansion of $\tan x$ in ascending powers of $\left(x - \frac{\pi}{4}\right)$.

Deduce that if $\left(x - \frac{\pi}{4}\right)$ is so small that $\left(x - \frac{\pi}{4}\right)^2$ and higher powers may be neglected, then $tan\, x \approx 1 - \frac{\pi}{2} + 2x$.

7.

a. Given that z_1 and z_2 are complex numbers, show geometrically, or otherwise that $|z_1| - |z_2| \leq |z_1 - z_2|$. Hence or otherwise, show that if z is a complex number such that $|z^2 - 3z| = 4e^{i\alpha}$, where α is real, then $|z| \leq 4$.

b. Given that $z = e^{i\theta}$, show that $z^n + \frac{1}{z^n} = 2\cos n\theta$.

Hence, show that

$$\frac{\cos 5\theta}{\cos\theta} = 16\sin^4\theta - 112\sin^2\theta + 1,$$

for $\cos\theta \neq 0$.

c. Show that the transformation $\omega = \frac{3z+6i}{iz-1}$ maps the line $|z+i| = |z+2i|$ to the curve $|\omega| = 3$.

8.

a. Solve for real x, in the equation $\sinh 2x - 2\cosh 2x + 2 = 0$.

b. Express $\tanh x$ in terms of e^x and e^{-x} and hence show that

$$\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}.$$

c. Given that

$$I_n = \int_0^{\frac{\pi}{4}} \tan^{\mathrm{n}} x\, dx,$$

show that $I_n + I_{n-2} = \frac{1}{n-1}$, $(n \geq 2)$.

Hence, find $\int_0^{\frac{\pi}{8}} \tan^4 2x\, dx$.

9.

a. Prove that if A and B are $n \times n$ non-singular matrices, then

$$(AB)^{-1} = B^{-1}A^{-1}.$$

b. Show that under the transformation represented by the matrix M, where

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & -2 \\ 3 & -2 & -7 \end{pmatrix},$$

the whole space is mapped onto the plane $x - 2y + z = 0$.

Find the image under this transformation of:

i. The line $x = -y = \frac{z}{2}$,

ii. The plane $x - y + z = 0$, giving your answer in Cartesian form.